

Hands on cusp package tutorial

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1 Introduction

The cusp package provides routines for fitting a cusp catastrophe model as suggested by (Cobb, 1978). The full documentation of the package can be found in the vignette “Fitting the Cusp Catastrophe in R: A **cusp**-Package Primer” included with the package.

2 Getting started

Load the library with the statement

```
> library(cusp)
```

3 The Zeeman data

We analyze three data sets that were obtained with three different instances of a Zeeman catastrophe machine. This device consists of a wheel is tethered by an elastic chord to a fixed point. For sake of reference let the *central axis* be defined as the axis parallel to the line between the fixed point and the center of the wheel. Another elastic is also attached to the wheel on one end, while the other end is moved about in the ‘control plane’ area opposite to the fixed point. The shortest distance between the strap point on the wheel and the central axis is recorded as a function of the position in the control plane. (In the original machine the angle between this axis and the line through the wheel center and the strap point is used.) See <https://www.math.sunysb.edu/~tony/whatsnew/column/catastrophe-0600/cusp4.html> for a vivid demonstration. The behavior of the Zeeman catastrophe machine is archetypal for the deterministic Cusp Catastrophe.

The data sets were obtained from 3 different physical instances of this machine, made by different people.

Measurements from this machine will be made with measurement errors, e.g., due to parallax, friction, writing mistakes etc. The measurements will fit the following model, the

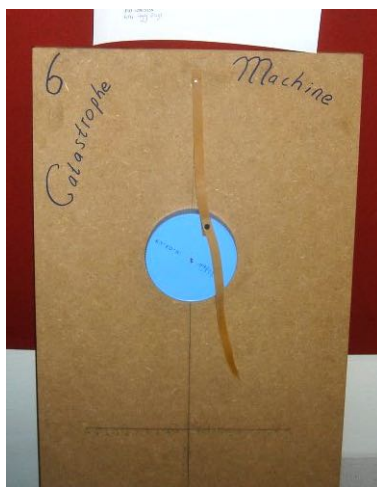


Figure 1: Catastrophe machine that was used to collect the data in `zeeman2`

we will dub the measurement error model,

$$y_i = z_i + \epsilon_i, \quad (1)$$

where ϵ_i is a zero mean random variable, e.g., $\epsilon \sim N(0, \sigma^2)$ for some σ^2 , and z_i is one of the extremal real roots of the cusp catastrophe equation

$$\alpha_i + \beta_i z - z^3 = 0.$$

Note that this is quite different from Cobb's conceptualization of the stochastic cusp catastrophe, in which

$$y_i \sim \Psi e^{\alpha_i y + \beta_i y^2 - \frac{1}{4} y^4}.$$

where the right hand side is the state density of the *stochastic* differential equation

$$dY = (\alpha_i + \beta_i z - z^3)dt + dW(t), \quad (2)$$

where $W(t)$ is a Wiener process.

The cusp package is intended for the latter model (i.e., for Cobb's stochastic catastrophe model). However, the data from the Zeeman catastrophe machine below show that it is also quite suitable for the former model.

3.1 `zeeman1`

The first data set

```
> data(zeeman1)
> nrow(zeeman1)
```

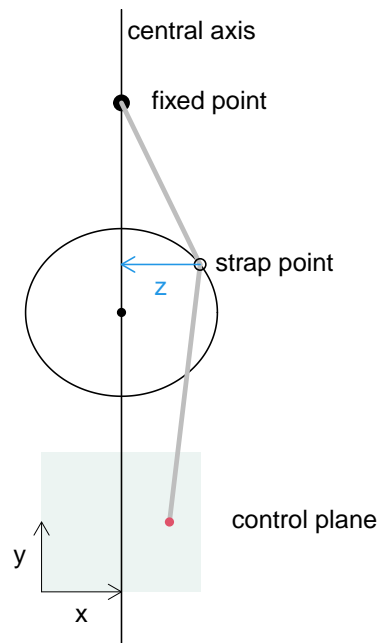


Figure 2: Catastrophe machine used for the data set `zeeman1` (left). Schematic Zeeman machine and the variables available in the data frames (right).

```
[1] 150
```

```
> head(zeeman1)
```

```
   x y   z
1 -7 2 -2.0
2 -6 2 -1.8
3 -5 2 -1.8
4 -4 2 -1.3
5 -3 2 -0.8
6 -2 2 -0.5
```

consists of 150 observations from experiments with the machine.

The columns of `zeeman1` respectively contain the value on the asymmetry axis which is orthogonal to the central axis (x), the value of the bifurcation variable which is parallel to the central axis (y), and state variable, the shortest distance from the wheel strap point to the central axis. Figure 2 displays these coordinates.

While thus the asymmetry and bifurcation variables are known, namely x and y respectively, we fit the model where each parameter is regressed on both of the control variables:

```
> fit1.1 = cusp(y~z, alpha~x+y, beta~x+y, zeeman1)
> summary(fit1.1)
```

Call:

```
cusp(formula = y ~ z, alpha = alpha ~ x + y, beta = beta ~ x +
      y, data = zeeman1)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-2.26705	-0.11605	-0.01376	0.10961	2.04443

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
a[(Intercept)]	0.21470	0.25516	0.841	0.40011
a[x]	1.15341	0.15558	7.414	1.23e-13 ***
a[y]	-0.16460	0.11996	-1.372	0.17002
b[(Intercept)]	1.04763	0.37401	2.801	0.00509 **
b[x]	0.02851	0.14058	0.203	0.83927
b[y]	-1.48202	0.13634	-10.870	< 2e-16 ***
w[(Intercept)]	-0.02387	0.13178	-0.181	0.85625
w[z]	0.90316	0.02723	33.168	< 2e-16 ***

```
---
```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance: 539.492 on 149 degrees of freedom
Linear deviance: 85.233 on 146 degrees of freedom
Logist deviance: NA on NA degrees of freedom
Delay deviance: 24.228 on 142 degrees of freedom

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.8711298	-170.44770	4	348.8954	349.1713	360.9379
Cusp model	0.9554786	76.23194	8	-136.4639	-135.4426	-112.3788

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 493.4, df = 4, p-value = 0

Number of optimization iterations: 72

The summary table indicates that the regression coefficients `a[(Intercept)]`, `a[y]`, `b[x]`, and `w[(Intercept)]` are non-significant suggesting that they are not required in the model. We therefore fit the model again, leaving these regressors out of the model:

```
> fit1.2 = cusp(y~z-1, alpha~x-1, beta~y, zeeman1)
> summary(fit1.2) # compare with logistic fit as well
```

Call:

```
cusp(formula = y ~ z - 1, alpha = alpha ~ x - 1, beta = beta ~
y, data = zeeman1)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-2.15809	-0.04248	0.04943	0.16451	2.11321

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
a[x]	1.15341	0.15540	7.422	1.15e-13	***
b[(Intercept)]	0.96994	0.37551	2.583	0.00979	**
b[y]	-1.47496	0.13656	-10.801	< 2e-16	***
w[z]	0.89487	0.02723	32.862	< 2e-16	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance: 529.634 on 149 degrees of freedom
Linear deviance: 85.233 on 146 degrees of freedom
Logist deviance: NA on NA degrees of freedom
Delay deviance: 23.365 on 146 degrees of freedom

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.8716421	-170.4477	4	348.8954	349.1713	360.9379
Cusp model	0.9568910	70.7295	4	-133.4590	-133.1831	-121.4165

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 482.4, df = 0, p-value = 0

Number of optimization iterations: 18

Now all the regression coefficients 'are significant'. If we turn attention to the information criteria however, we see that the AIC for the first model (-136.464) is smaller than the AIC for the second model (-133.459). The BIC on the other hand, is larger for the first model (-112.379) compared to the second model (-133.459). Because in the first model especially the intercept for that state model equation is quite small ($w[(\text{Intercept})] = -0.0239$), we relax the second model a bit by allowing for intercept in model equation for α to see if the AIC is smaller

```
> fit1.3 = cusp(y~z-1, alpha~x, beta~y, zeeman1)
> (sf1.3 <- summary(fit1.3, logist=TRUE))
```

```
Call:
cusp(formula = y ~ z - 1, alpha = alpha ~ x, beta = beta ~ y,
      data = zeeman1)
```

```
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.21248 -0.10346  0.00771  0.12443  2.07045
```

```
Coefficients:
```

	Estimate	Std. Error	z value	Pr(> z)	
a[(Intercept)]	0.45555	0.17266	2.638	0.00833	**
a[x]	1.15341	0.15471	7.455	8.98e-14	***
b[(Intercept)]	0.99190	0.37313	2.658	0.00785	**
b[y]	-1.47891	0.13597	-10.877	< 2e-16	***
w[z]	0.89766	0.02711	33.114	< 2e-16	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance: 532.944 on 149 degrees of freedom
Linear deviance: 85.233 on 146 degrees of freedom
Logist deviance: 26.145 on 146 degrees of freedom
Delay deviance: 24.683 on 145 degrees of freedom

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.8716421	-170.44770	4	348.8954	349.1713	360.9379
Logist model	0.9604700	-81.81669	4	171.6334	171.9092	183.6759
Cusp model	0.9540613	74.72029	5	-139.4406	-139.0239	-124.3874

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 490.3, df = 1, p-value = 0

Number of optimization iterations: 31

The AIC, AICc and BIC are now all smaller than both the first and the second model, and all intercept coefficients 'are significant'.

In the call to `summary`, this time we have set the optional parameter `logist` to `TRUE`, so that also the logistic model is fitted for a more critical test of the cusp model (see the main vignette and `help(cusp.logist)` for details). The information criteria all indicate that the cusp model is to be preferred over the logistic model, even though the *pseudo-R²* is larger for the logistic model (0.9605) than for the cusp model (0.9541). (See for a discussion of the *pseudo-R²* the main vignette.)

Figure 3 displays a diagnostic plot of the last model that was generated with

```
> plot(fit1.3)
```

```
> plot(fit1.3)
```

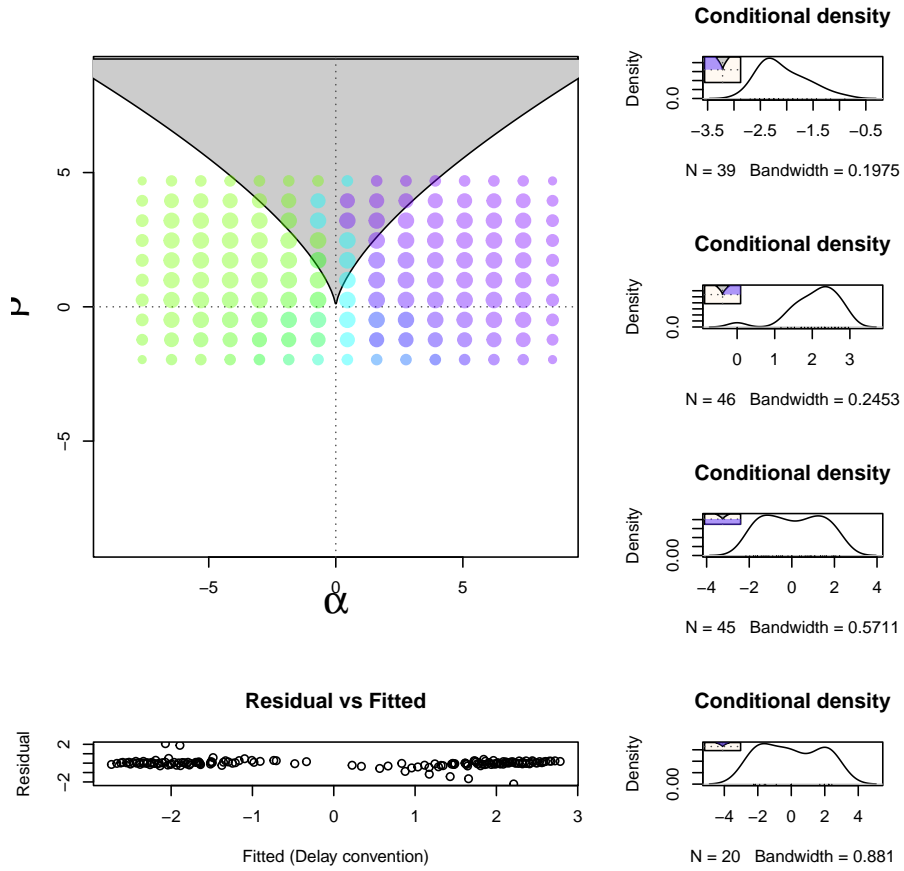


Figure 3: Diagnostic plot for the second model fitted to `zeeman1`.

3.2 zeeman2

We fit the same models as in the previous case

```
> data(zeeman2)
> fit2.1 = cusp(y~z, alpha~x+y, beta~x+y, zeeman2)
> summary(fit2.1)
```

Call:

```
cusp(formula = y ~ z, alpha = alpha ~ x + y, beta = beta ~ x +
      y, data = zeeman2)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-1.589e+00	-1.839e-01	-3.037e-06	1.839e-01	1.589e+00

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
a[(Intercept)]	-1.331e-04	2.084e-01	-0.001	0.99949
a[x]	7.892e-01	9.925e-02	7.952	1.84e-15 ***
a[y]	1.433e-05	4.994e-02	0.000	0.99977
b[(Intercept)]	3.891e+00	2.864e-01	13.588	< 2e-16 ***
b[x]	1.154e-05	4.585e-02	0.000	0.99980
b[y]	1.007e-01	3.448e-02	2.921	0.00349 **
w[(Intercept)]	8.802e-06	4.799e-02	0.000	0.99985
w[z]	7.789e-01	1.791e-02	43.495	< 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance:	1100.117	on 197	degrees of freedom
Linear deviance:	426.846	on 194	degrees of freedom
Logist deviance:	NA	on NA	degrees of freedom
Delay deviance:	25.421	on 190	degrees of freedom

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.7645990	-356.9972	4	721.9944	722.2016	735.1475
Cusp model	0.9770089	121.4310	8	-226.8621	-226.1002	-200.5559

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 956.9, df = 4, p-value = 0

Number of optimization iterations: 94

Again, the intercept coefficients a[(Intercept)] and w[(Intercept)] are non-significant, as are a[y] and b[x]. We leave these coefficients out of the model in the followup fit

```
> fit2.2 = cusp(y~z-1, alpha~x-1, beta~y, zeeman2)
> summary(fit2.2)
```

Call:

```
cusp(formula = y ~ z - 1, alpha = alpha ~ x - 1, beta = beta ~
      y, data = zeeman2)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-1.589e+00	-1.839e-01	3.331e-15	1.839e-01	1.589e+00

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
a[x]	0.78913	0.09924	7.952	1.84e-15	***
b[(Intercept)]	3.89115	0.28636	13.588	< 2e-16	***
b[y]	0.10071	0.03448	2.921	0.00349	**
w[z]	0.77891	0.01791	43.495	< 2e-16	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance: 1100.104 on 197 degrees of freedom
 Linear deviance: 426.846 on 194 degrees of freedom
 Logist deviance: NA on NA degrees of freedom
 Delay deviance: 25.421 on 194 degrees of freedom

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.7645990	-356.9972	4	721.9944	722.2016	735.1475
Cusp model	0.9770089	121.4310	4	-234.8621	-234.6548	-221.7090

Note: R.Squared for cusp model is Cobb's pseudo-R^2. This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 956.9, df = 0, p-value = 0

Number of optimization iterations: 34

The AIC of this reduced model (-234.8621) is indeed smaller than the AIC of the previous model (-226.8621). If for comparison with the model selected for `zeeman1` we include `a[(Intercept)]` again, this barely reduces the negative log-likelihood, while increasing the AIC. More importantly this coefficient is estimated to be zero.

```
> fit2.3 = cusp(y~z-1, alpha~x, beta~y, zeeman2)
> fit2.3
```

```
Call: cusp(formula = y ~ z - 1, alpha = alpha ~ x, beta = beta ~ y, data = zeeman2)
```

Coefficients:

a[(Intercept)]	a[x]	b[(Intercept)]	b[y]	w[z]
1.008e-16	7.891e-01	3.891e+00	1.007e-01	7.789e-01

Degrees of Freedom: 197 Total (i.e. Null); 193 Residual

Null Deviance: 1100

Delay Deviance: 25.42 AIC: -232.9

3.3 zeeman3

We do the same analysis for `zeeman3`.

```
> data(zeeman3)
> fit3.1 <- cusp(y~z, alpha~x+y, beta~x+y, zeeman3)
> summary(fit3.1)
```

Call:

```
cusp(formula = y ~ z, alpha = alpha ~ x + y, beta = beta ~ x +
y, data = zeeman3)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-1.28904	-0.14995	0.03348	0.17721	2.44220

Coefficients:

Estimate	Std. Error	z value	Pr(> z)
----------	------------	---------	----------

```

a[(Intercept)] -0.0540490  0.0033964  -15.914   <2e-16 ***
a[x]           0.2437644  0.0006382  381.935   <2e-16 ***
a[y]          -0.0185760  0.0008130  -22.848   <2e-16 ***
b[(Intercept)]  4.8832846  0.0034005 1436.034   <2e-16 ***
b[x]          -0.0454644  0.0006383  -71.223   <2e-16 ***
b[y]          -0.6766923  0.0008136 -831.715   <2e-16 ***
w[(Intercept)] -0.0564834  0.0219623   -2.572    0.0101 *
w[z]           0.4664489  0.0044566  104.666   <2e-16 ***

```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

Null deviance: 987.386 on 281 degrees of freedom
Linear deviance: 2016.486 on 278 degrees of freedom
Logist deviance:      NA on  NA degrees of freedom
Delay deviance:  55.197 on 274 degrees of freedom

```

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.5556587	-677.51652	4	1363.03304	1363.17744	1377.6007
Cusp model	0.9451417	-34.08026	8	84.16052	84.68799	113.2958

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 1287, df = 4, p-value = 0

Number of optimization iterations: 114

This time all regression coefficients are significant. We still compare with the simplified model used in the other cases to see if the AIC, AICc and BIC might be lower still

```

> fit3.2 <- cusp(y~z-1, alpha~x-1, beta~y, zeeman3)
> summary(fit3.2)

```

Call:

```

cusp(formula = y ~ z - 1, alpha = alpha ~ x - 1, beta = beta ~
y, data = zeeman3)

```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-----	----	--------	----	-----

-2.496125 -0.182561 -0.005285 0.158835 1.054552

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
a[x]	0.24273	0.02918	8.317	<2e-16 ***
b[(Intercept)]	4.87001	0.25664	18.976	<2e-16 ***
b[y]	-0.68031	0.04054	-16.781	<2e-16 ***
w[z]	0.46586	0.01051	44.334	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance:	984.900	on 281	degrees of freedom
Linear deviance:	2016.486	on 278	degrees of freedom
Logist deviance:	NA	on NA	degrees of freedom
Delay deviance:	56.263	on 278	degrees of freedom

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.5563031	-677.51652	4	1363.03304	1363.17744	1377.60067
Cusp model	0.9445425	-36.21439	4	80.42879	80.57319	94.99642

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 1283, df = 0, p-value = 0

Number of optimization iterations: 25

Compared to the previous fit, this model indeed yields a lower AIC, AICc, and BIC and hence is preferred according to the information criteria.

4 Simulation

Sometimes it's useful to be able to simulate data from a model. Two cases are discussed here.

4.1 Generating data in accordance with Cobb's cusp SDE

To obtain observations in accordance with Cobb's stochastic cusp catastrophe model in equation (2), the function `rcusp` can be used. If for all the observations α and β are equal, this is simply done with the statement. Figure 4 displays a histogram.

```
> set.seed(423)
> alpha = 0.25
> beta = 2
> n = 1000
> y = rcusp(n, alpha, beta)
> hist(y,80,freq=FALSE)
> curve(drcusp(x, 0.25, 2), min(y)-1, max(y)+1, add=TRUE, col=2)
```

The parameters α and β can simply be estimated with the statement

```
> cusp(y~y-1, alpha~1, beta~1)
```

```
Call: cusp(formula = y ~ y - 1, alpha = alpha ~ 1, beta = beta ~ 1)
```

Coefficients:

a[(Intercept)]	b[(Intercept)]	w[y]
0.1916	2.0654	1.0081

Degrees of Freedom: 999 Total (i.e. Null); 997 Residual

Null Deviance: 1641

Delay Deviance: 361.2 AIC: 2278

If the α 's and β 's are different, e.g., depending on predictive variables, they can be generated for example with the statements

```
> set.seed(423)
> x1 = runif(150)
> x2 = runif(150)
> a = c(-2, 4)
> b = c(-1, 4)
> alpha = a[1] + a[2]*x1
> beta = b[1] + b[2]*x2
> z = Vectorize(rcusp)(1, alpha, beta)
> data <- data.frame(x1, x2, z)
```

Estimating the coefficients in `a` and `b` can then be carried out with, e.g.,

```
> fit <- cusp(y ~ z, alpha ~ x1+x2, beta ~ x1+x2, data)
```

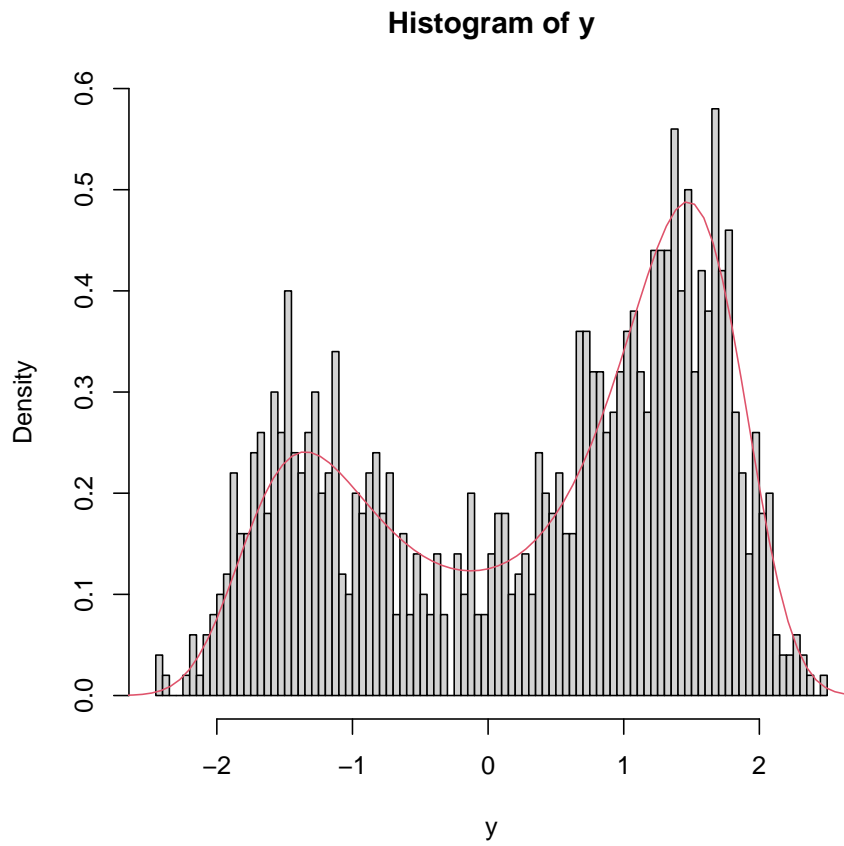


Figure 4: Histogram of 1000 observations from Cobb's stochastic cusp catastrophe SDE (as opposed to the deterministic cusp catastrophe) with for each observation $\alpha = 0.25$ and $\beta = 2$.

4.2 Generating data in accordance with the measurement error model

We can generate data in accordance with the measurement error model in equation (1) as follows

```
> set.seed(423)
> g = expand.grid(seq(-3,3,len=15), seq(-3,3,len=15))
> a = g[,1]; b = g[,2]; idx=cbind(sample(c(1,3),length(a),TRUE), seq(along=a));
> s = Vectorize(cusp.extrema)(a,b)[idx]
> y = s + rnorm(length(s),,.3)
```

Here `g` is a grid of points on the control surface, and `cusp.extrema` is used to compute the roots of the cusp equilibrium equation

$$\alpha + \beta y^2 - y^3 = 0.$$

The vector of states `s` is then constructed by sampling from the smallest and largest roots. The observations `y` are then created in accordance with the measurement error model.

It instructive to see a 3D plot of the generated data points generated in this way

```
> if(require(plot3D)){
+   scatter3D(a, b, y, theta=200, phi=10, zlim=c(-4,4));
+   # you can try require(rgl); rgl.points(a, b, y) instead
+ }
```

Note that this measurement model is *not* cusp SDE model of Cobb as described in the paper (the main vignette that comes with the cusp package). Even so, the maximum likelihood method of Cobb, does fit this surface quite well:

```
> fit <- cusp(y~y, alpha~a, beta~b)
> summary(fit)
```

Call:

```
cusp(formula = y ~ y, alpha = alpha ~ a, beta = beta ~ b)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-0.80324	-0.19146	0.02445	0.26135	2.81930

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
a[(Intercept)]	-0.144327	0.167897	-0.860	0.390
a[a]	1.620453	0.184198	8.797	<2e-16 ***
b[(Intercept)]	-0.127060	0.397925	-0.319	0.749

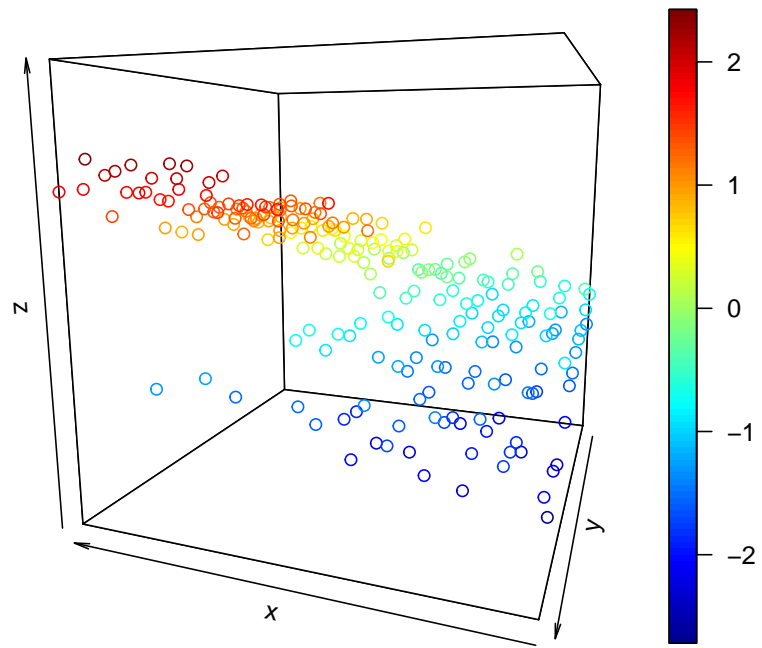


Figure 5: 3D scatter plot of cusp equilibrium surface with measurement error

```

b[b]          1.405980   0.101885  13.800   <2e-16 ***
w[(Intercept)] -0.009536   0.037232  -0.256    0.798
w[y]          1.122367   0.034857  32.199   <2e-16 ***

```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

Null deviance: 472.219 on 224 degrees of freedom
Linear deviance: 118.853 on 221 degrees of freedom
Logist deviance:      NA on  NA degrees of freedom
Delay deviance:  40.432 on 219 degrees of freedom

```

	R.Squared	logLik	npar	AIC	AICc	BIC
Linear model	0.6829434	-247.46224	4	502.9245	503.1063	516.5889
Cusp model	0.9160938	-54.00909	6	120.0182	120.4035	140.5148

Note: R.Squared for cusp model is Cobb's pseudo-R². This value can become negative.

Chi-square test of linear vs. cusp model

X-squared = 386.9, df = 2, p-value = 0

Number of optimization iterations: 66